NON-LINEAR HARMONIC MODELS FOR GEODETIC TIME SERIES

Modelos armónicos no lineales para series temporales geodéticas

Pedro Antonio Martínez Ortiz
Dept. Applied Mathematics
University of Alicante. Spain

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Memoria para la obtención del grado de Doctor

Director: José Manuel Ferrándiz Leal

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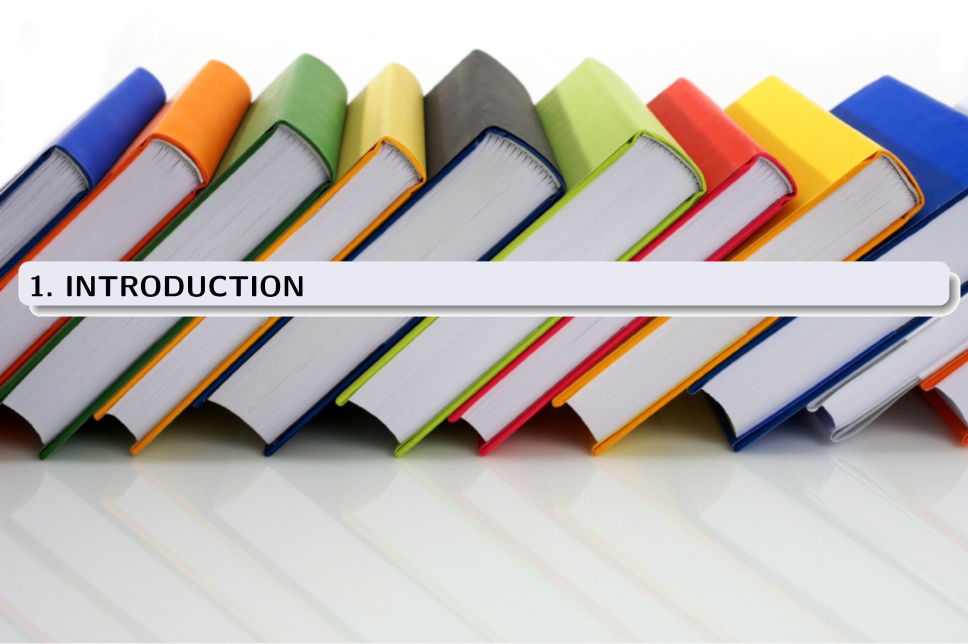

1. INTRODUCTION

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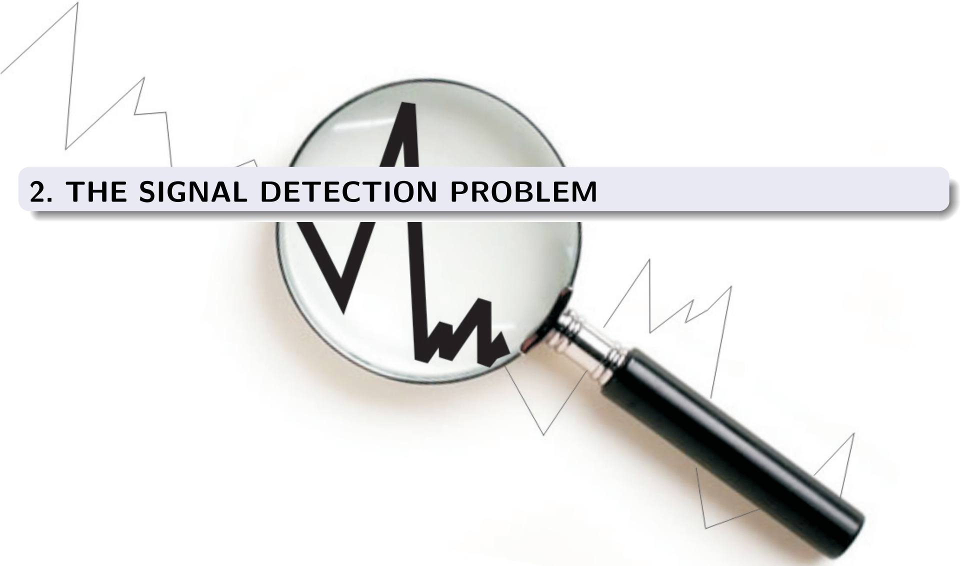
- We **have not focussed on the origin** or cause of the periodic patterns. We will suggest possible **candidate factors**.
- **MATLAB** was chosen as the main tool to carry out this task.



2. THE SIGNAL DETECTION PROBLEM

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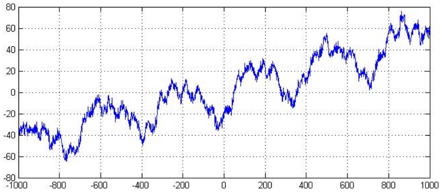

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- Short Periodic $\Rightarrow S(t)$
- Long Periodic $\Rightarrow L(t)$
- Periodic $\Rightarrow S(t) + L(t)$
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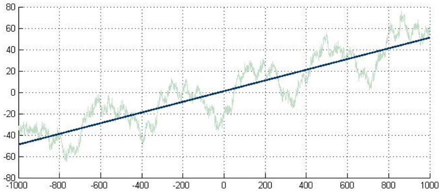

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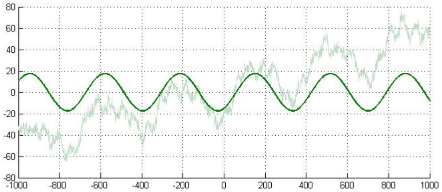

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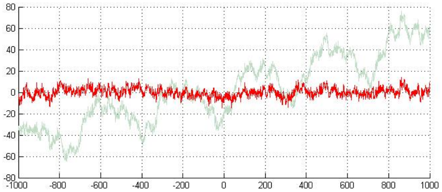

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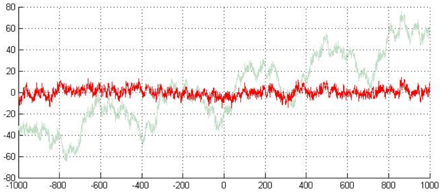

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- If it depends on two or more parameters $\Rightarrow$ **multidimensional**.

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- **PERIOD**, Π: time took by the phenomenon to complete a cycle.

$$\Pi = \frac{1}{f}$$

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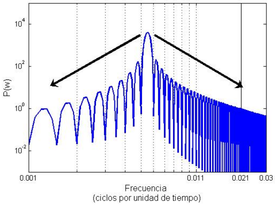

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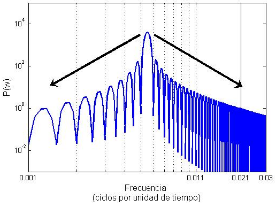
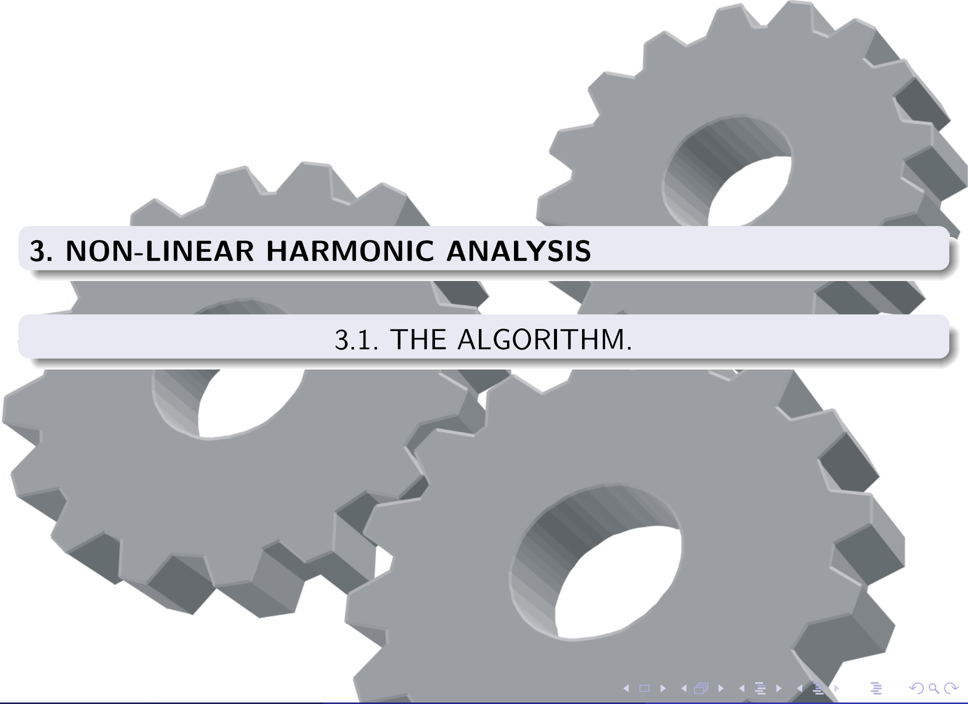
Solution

- You can avoid the inclusion of these spurious frequencies by drawing the harmonics from the periodogram **iteratively**.





3. NON-LINEAR HARMONIC ANALYSIS

3.1. THE ALGORITHM.

3.1. The algorithm: Non-linear harmonic analysis

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- **Frequencies** and other non-linear parameters (**BFGS**)
- **Number of non-linear parameters** (until the weighted residual root mean square, **WRMS**, turns out to be less than a threshold)

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- a_{lm} are the linear coefficients that need to be estimated.

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$$\psi_1(\tau) = 1, \qquad \psi_2(\tau) = \frac{4\tau}{T}, \qquad \psi_3(\tau) = \left(\frac{3(N-1)}{4(N+1)}\right)\psi_2^2(\tau) - 1$$

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where $T = t_N - t_1$, and $\tau = t - \frac{t_1 + t_N}{2} = t - t_1 - \frac{T}{2}$

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- Two **Fourier terms** for each $\{\omega_k, \;\; k = 1 \ldots K\}$

$$\psi_{2k+2}(\tau) = \sin(\omega_k \tau),$$
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- Two **mixed secular terms** for some frequencies $\{\omega_{k_i}\}$ for $i = 1 \dots S$.

$$\psi_{2K+(i+3)}(\tau) = \tau \sin(\omega_{k_i} \tau)$$
$$\psi_{2K+(i+4)}(\tau) = \tau \cos(\omega_{k_i} \tau)$$

3.1. The algorithm: Frequency extracting process

Lomb Periodogram

$$P(\omega) = \sum_{m=1}^{M} \left[\frac{X_m V_m^2 + Y_m U_m^2 - 2 Z_m U_m V_m}{X_m Y_m - Z_m^2} \right]$$

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$$P = Max\ P(\omega), \qquad \omega_P = Arg\ max\ P(\omega)$$

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$$Q(\omega) = \sum_{m=1}^{M} \left[\frac{X'_m {V'_m}^2 + Y'_m {U'_m}^2 - 2Z'_m U'_m V'_m}{X'_m Y'_m - {Z'_m}^2} \right]$$

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$$Z'_m = \sum_{n=1}^{N} \mu_{nm} \tau_n^2 \sin(\omega \tau_n) \cos(\omega \tau_n), \quad U'_m = \sum_{n=1}^{N} \mu_{nm} e_{nm} \tau_n \sin(\omega \tau_n),$$

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3.1. The algorithm: Frequency extracting process

START

Extended Periodogram

$$Q(\omega) = \sum_{m=1}^{M} \left[\frac{X'_m {V'_m}^2 + Y'_m {U'_m}^2 - 2 Z'_m U'_m V'_m}{X'_m Y'_m - {Z'_m}^2} \right]$$

$$X'_m = \sum_{n=1}^{N} \mu_{nm} \tau_n^2 \sin^2(\omega \tau_n), \quad Y'_m = \sum_{n=1}^{N} \mu_{nm} \tau_n^2 \cos^2(\omega \tau_n),$$

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$$Q = Max\ Q(\omega), \qquad \omega_Q = Arg\ max\ Q(\omega)$$

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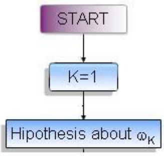
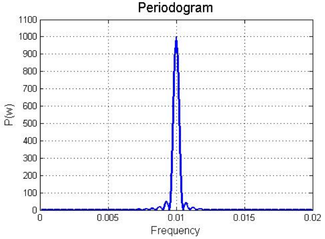
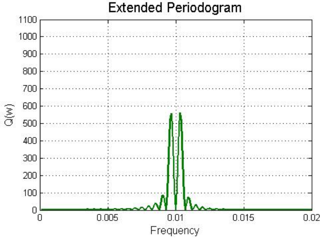
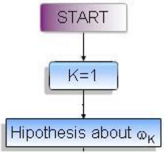


$$d(t) = \sin\left(2\pi \cdot 0.01 \cdot t\right)$$





- If $P > Q \Rightarrow \omega_k = \omega_P$ (**Fourier**)



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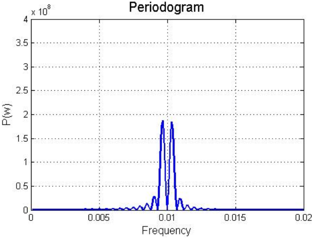
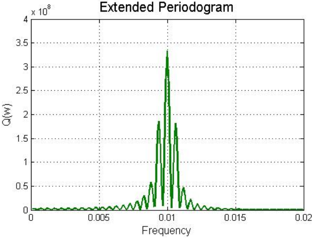
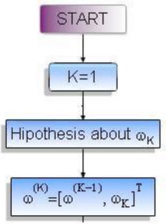
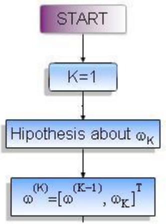




- If $P > Q \Rightarrow \omega_k = \omega_P$ (**Fourier**)
- If $P < Q \Rightarrow \omega_k = \omega_Q$ (**Fourier + secular**)



- We need to estimate the coefficients $\{a_{lm},\ l = 1, \dots, L\ ,\ m = 1, \dots, M\}$ that minimize the objective function $\phi(\vec{a}, \vec{\omega})$



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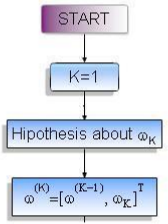
$\Rightarrow$ Least Square Method **(LSM)**

- Once we have $\vec{a} = \hat{a}(\vec{\omega})$, we can regard



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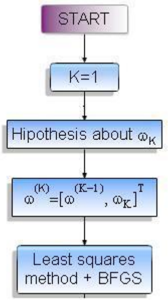

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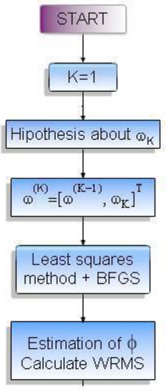
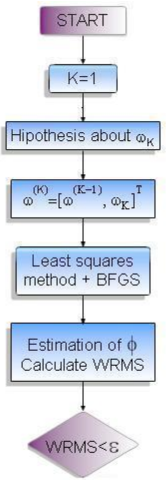
$\Rightarrow$ Quasi-Newton algorithm **BFGS**

3.1. The algorithm: Stop criteria



- Weighted residual root mean square, **WRMS**:

$$WRMS = \sqrt{\frac{\phi}{\sum_{n=1}^{N} \sum_{m=1}^{M} \mu_{nm}}}$$



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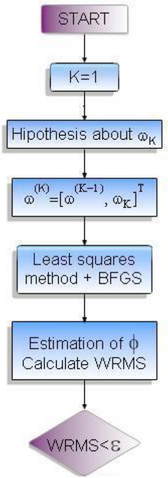

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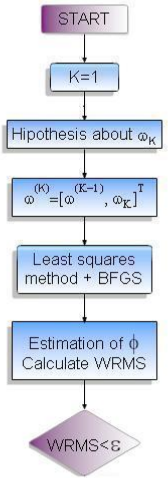

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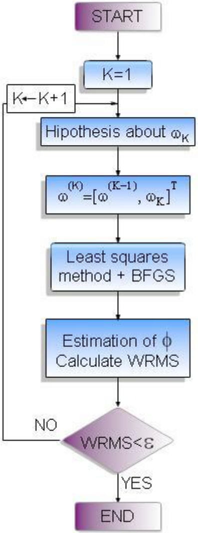

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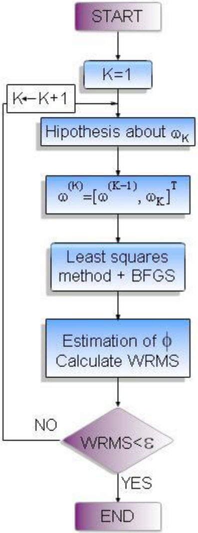

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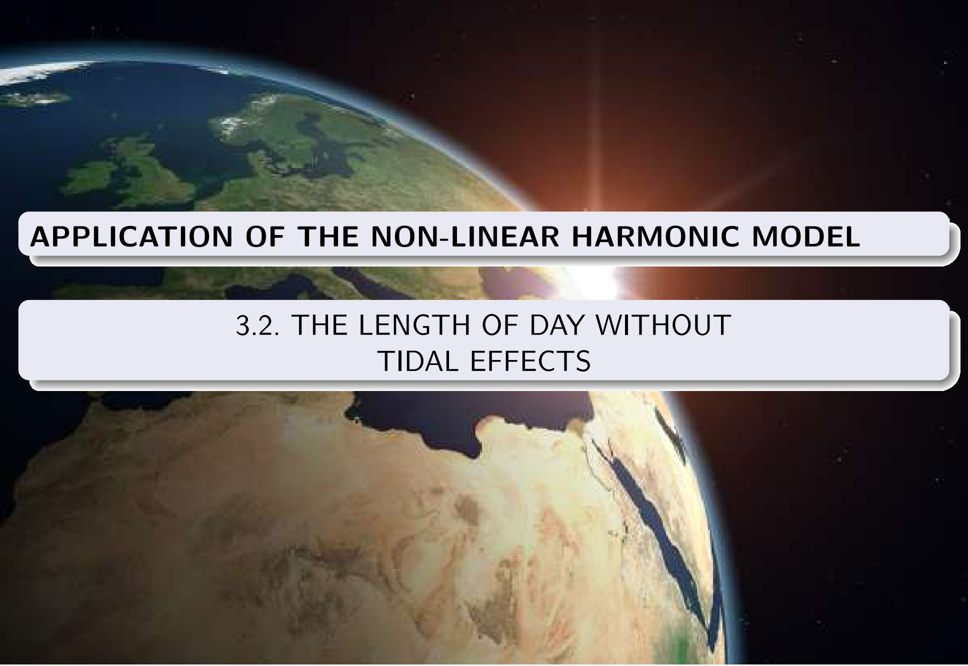
TO TAKE INTO ACCOUNT

The method estimates linear and non-linear parameters again and again as a new frequency is added to the model.



APPLICATION OF THE NON-LINEAR HARMONIC MODEL

3.2. THE LENGTH OF DAY WITHOUT TIDAL EFFECTS

3.2. Length of day without tidal effects

RAW DATA DESCRIPTION

- **Definition**: The variation of LOD (Δ) is the difference between the astronomically duration of the day and 86400 s.
- **Source**: VLBI data provided by LAREG at IGN (France)
- **Time domain**: Unevenly and daily time series from April 12th, 1980 to December 31st, 2008.
- **Number of observations**: 3197 $\Rightarrow$ 30 % of 28 years.

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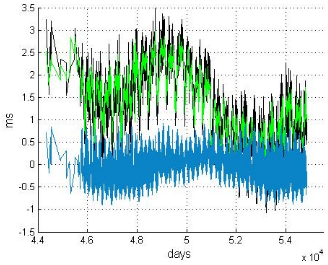


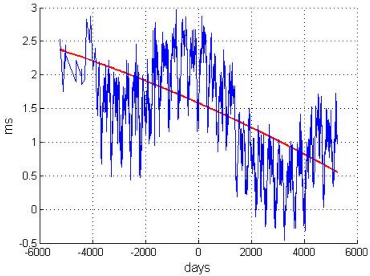
INPUT DATA

Tidal effects $\delta\Delta$ can be modeled with high accuracy (Petit and Luzum, 2010):

$$\widehat{\Delta} = \Delta - \delta\Delta$$

ANALYSIS DESCRIPTION AND RESULTS

- **Trend**: Quadratic polynomial trend component.
- **Periodogram step-size**: 10^{-5} cpd
- **Number of frequencies**: 15
- **Final WRMS**: 0.1839 ms
- **WRMS reduction**: 75.16 %



$$T_{LOD}(\tau_n) = (1.53931 \pm 0.00318) +$$
$$+ (-0.45459 \pm 0.00297)\,\varphi_2(\tau_n) +$$
$$+ (-0.03832 \pm 0.00406)\,\varphi_3(\tau_n)$$

$$\tau_n = t_n - 49586.5 \ \mathrm{MJD}$$

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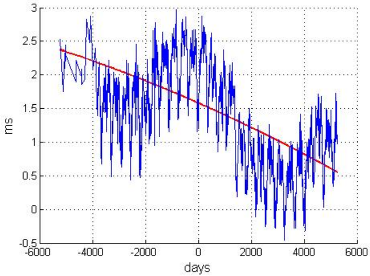

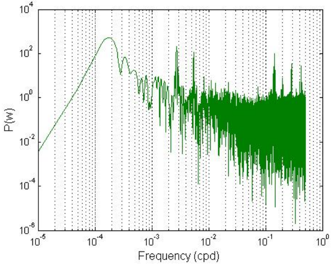

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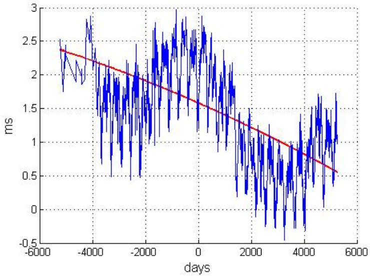

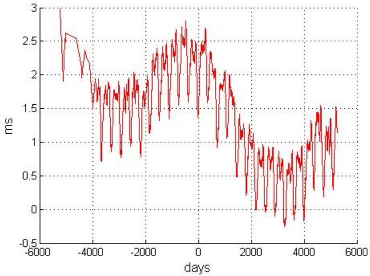

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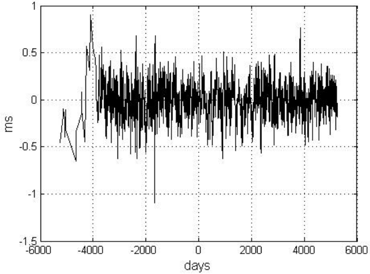

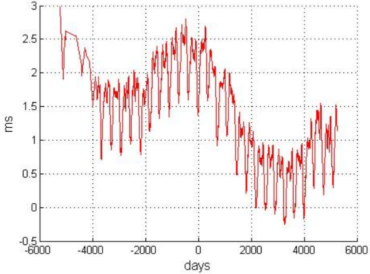

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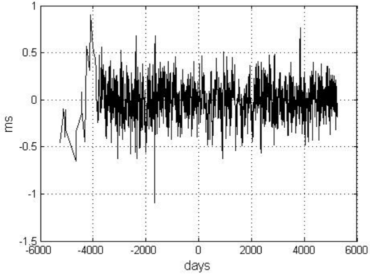

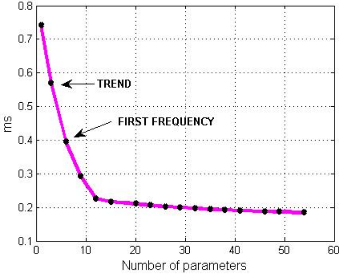

3.2. Length of day without tidal effects

Π	S	C
days	ms	ms
5916.35	-0.1247	0.6172
365.07	0.0931	-0.3455
182.73	0.1499	-0.2115
870.38	0.0893	0.0040
1631.87	-0.0006	-0.0050
981.52	0.0495	-0.0293
121.84	0.0419	0.0211
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- 16.20 years with an amplitude of 0.6296 ± 0.0037 *ms*
- Annual signal with an amplitude of 0.3578 ± 0.0033 *ms*
- Semiannual signal with an amplitude of 0.2593 ± 0.0033 *ms*
- Lunar semi-sidereal period
- Periods ranging from 1.8 to 2.5 years $\Rightarrow$ **QBO?**
- Period of ≈ 4.47 years $\Rightarrow$ **ENSO?**Period is linked to mixed secular terms with an amplitude of 0.0288 μs.
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1631.87	-0.0006	-0.0050
981.52	0.0495	-0.0293
121.84	0.0419	0.0211
690.65	-0.0027	-0.0511
91.05	-0.0141	0.0424
13.66	0.0229	-0.0386
780.28	-0.0048	-0.0473
125.67	0.0322	0.0234
98.78	0.0037	-0.0070
75.18	-0.0037	-0.0379
71.79	-0.0057	0.0165

CONCLUSIONS

- 16.20 years with an amplitude of 0.6296 ± 0.0037 ms
- Annual signal with an amplitude of 0.3578 ± 0.0033 ms
- Semiannual signal with an amplitude of 0.2593 ± 0.0033 ms
- Lunar semi-sidereal period
- Periods ranging from 1.8 to 2.5 years $\Rightarrow$ **QBO?**
- Period of $\approx$ 4.47 years $\Rightarrow$ **ENSO?**Period is linked to mixed secular terms with an amplitude of 0.0288 μs.
- Also linked to mixed secular terms.

Π	S	C
days	ms	ms
5916.35	-0.1247	0.6172
365.07	0.0931	-0.3455
182.73	0.1499	-0.2115
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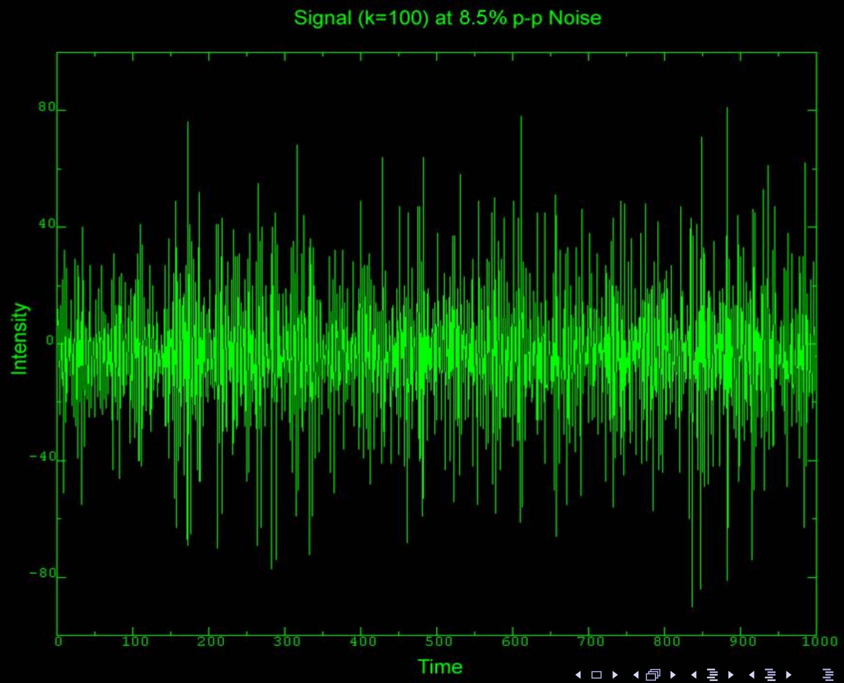
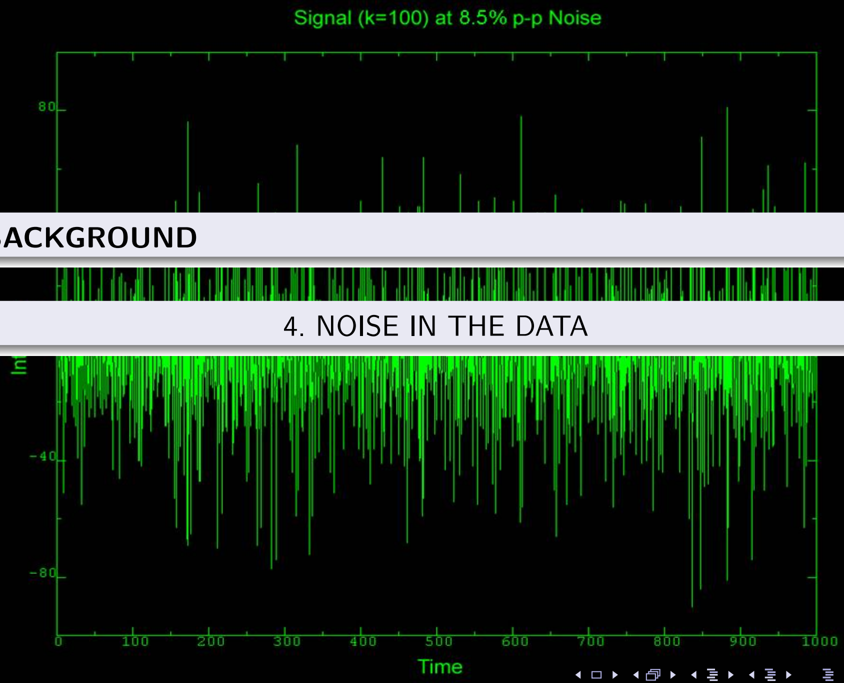

Signal (k=100) at 8.5% p-p Noise
Intensity
80
40
0
-40
-80
0
100
200
300
400
500
600
700
800
900
1000
Time



BACKGROUND

4. NOISE IN THE DATA

4. Background: noise in the data

POWER SPECTRA

$$P(f) = P_0 \cdot f^k$$

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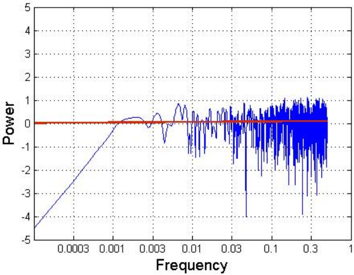

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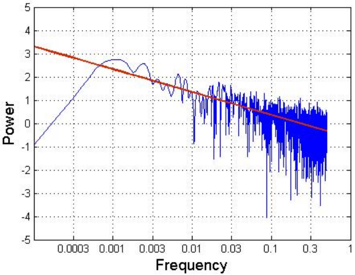

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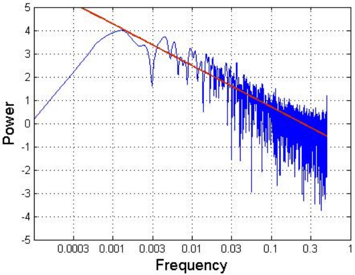

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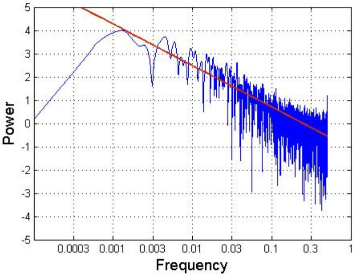

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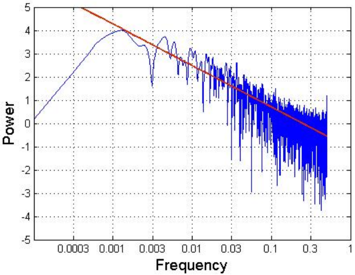

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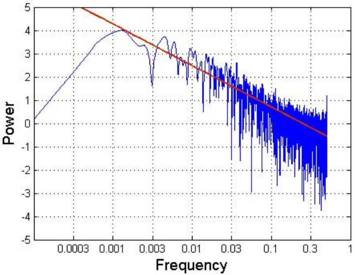

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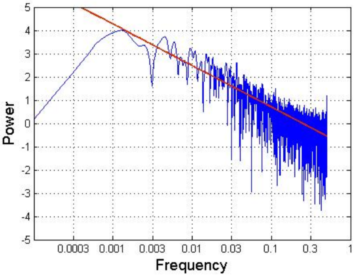

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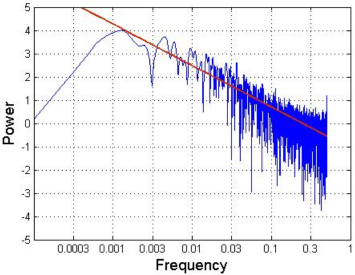

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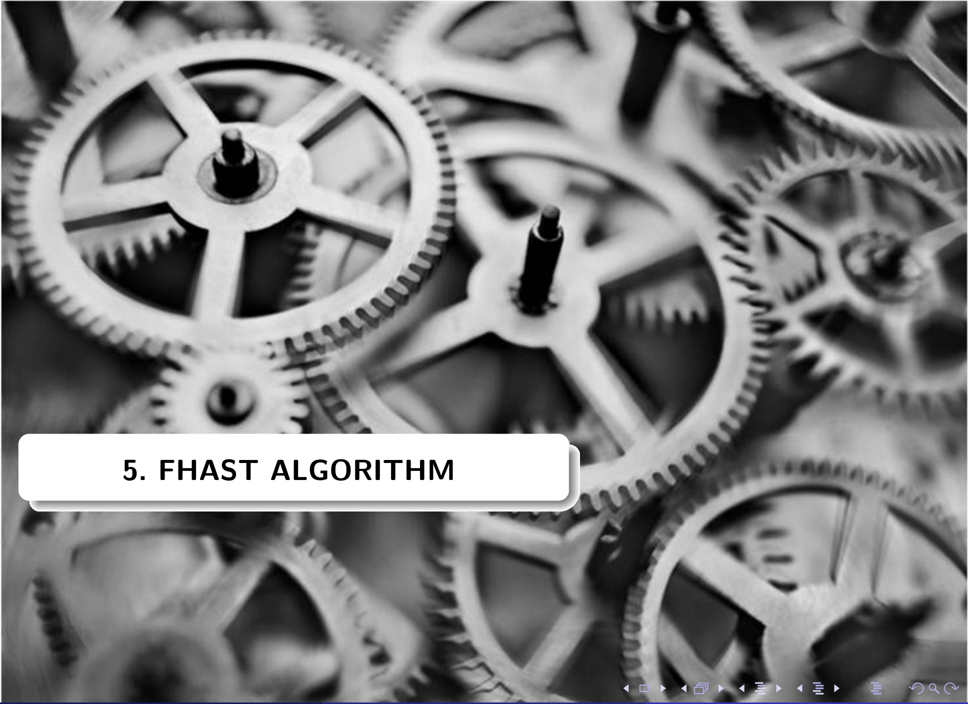

5. FHAST ALGORITHM

This **new algorithm** combines the strongest points of:

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We **avoid** including **spurious frequencies** that might refer to the noise but not to the phenomenon represented by the data.

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Simplification: **Scalar** time series + **single colored noise** component

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Objective function for white noise component

$$\phi = \sum_{n=1}^{N} \mu_n \left[d_n - \left\{ \sum_{l=1}^{L} a_l \psi_l(t_n) \right\} \right]^2 = \sum_{n=1}^{N} \mu_n \left(d_n - h_n \right)^2$$

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TREND

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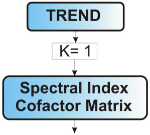

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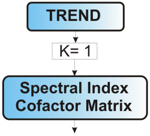

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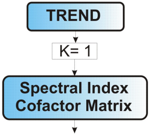

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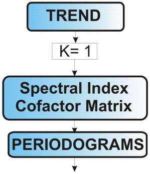

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$$P\left(B_j, \omega_j\right) = d^{\,T} Q_y^{-1} B_j \left(B_j^{\,T} Q_y^{-1} B_j\right)^{-1} B_j^{\,T} Q_y^{-1} d$$

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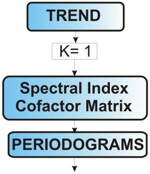

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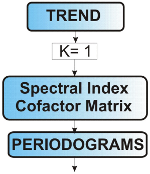
Denoised Lomb Periodogram

$$B_j = \begin{pmatrix} \sin(\omega_j t_1) & \cos(\omega_j t_1) \\ \sin(\omega_j t_2) & \cos(\omega_j t_2) \\ \vdots & \vdots \\ \sin(\omega_j t_N) & \cos(\omega_j t_N) \end{pmatrix}$$



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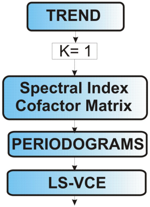

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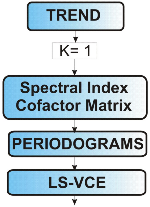

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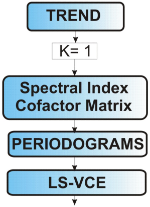

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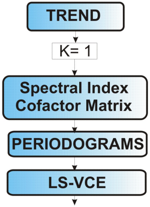

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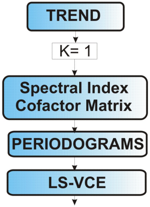

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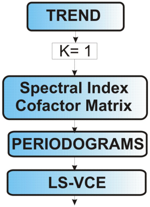

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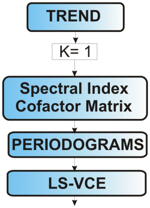

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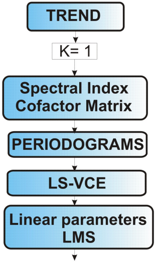
$$\hat{\sigma}^{i+1} = G^{-1} v$$

Step 5: Estimate linear and non-linear parameters of the functional model by using **LSM** and **BFGS**.



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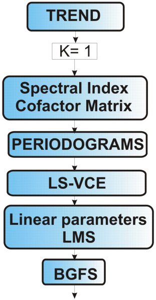

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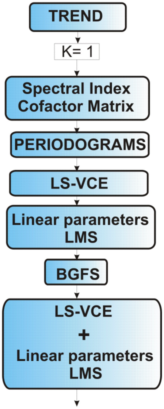

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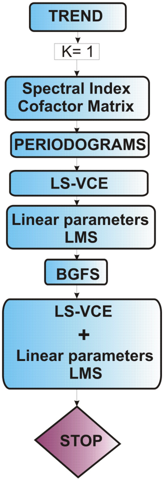

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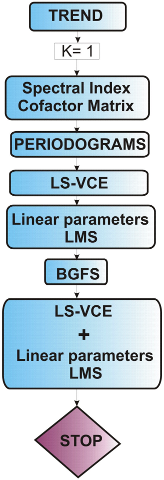
STOP CRITERIA

- WRMS$< \epsilon$



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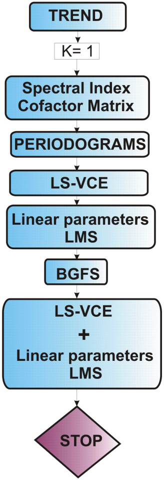

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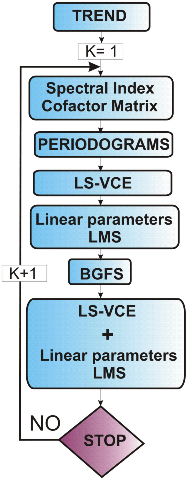

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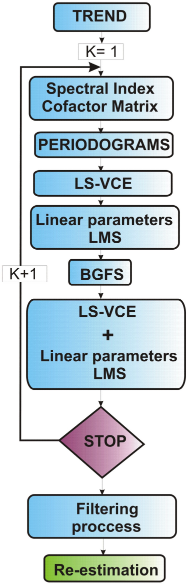
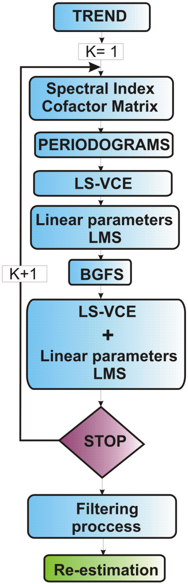


FILTERING PROCESS

- Frequencies with a **SNR** less than certain value are **removed** and the model is **re-estimated** with the **remaining signals**.



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IMPORTANT

- All the **parameters** in the functional model, as well as the variance component, are **re-estimated** again and again **as a new frequency is added** to the model.

5.2. Synthetic example

SYNTHETIC SERIES

- Includes 1.04, 3 and 4.16 cycles/year (the latter linked to mixed secular terms).
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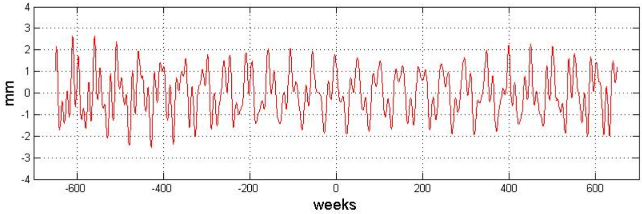

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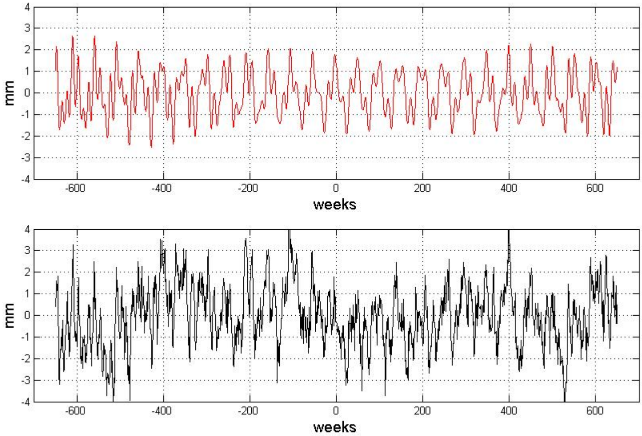
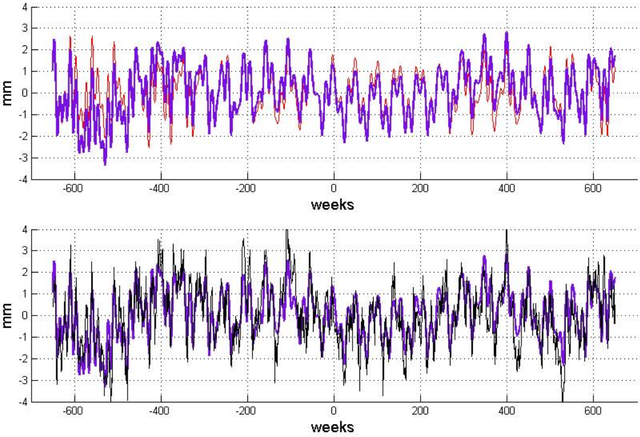


NON-LINEAR HARMONIC ALGORITHM

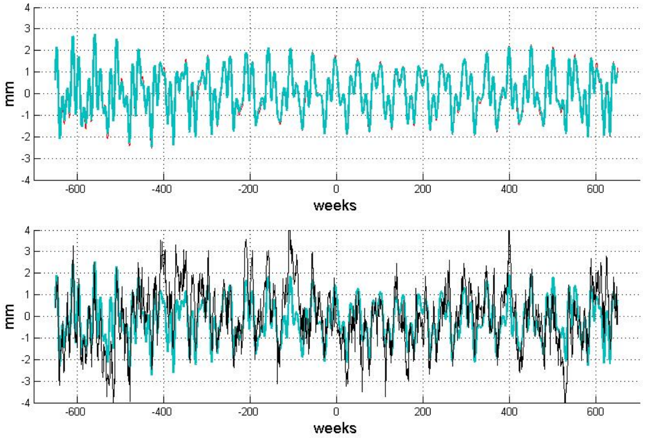
- Finds frequencies that not explain the nature of the time serie.
- Does not estimate the noise component.



5.2. Synthetic example

FHAST ALGORITHM

- Gets the real frequencies with the correct amplitude and basis functions.
- Estimates the noise component properly.



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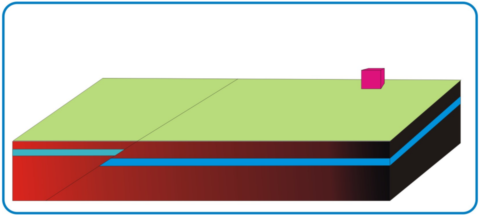

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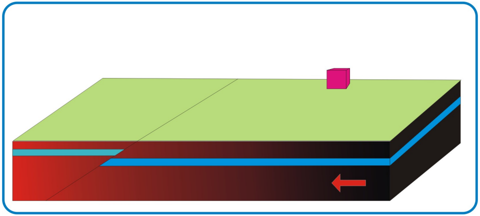

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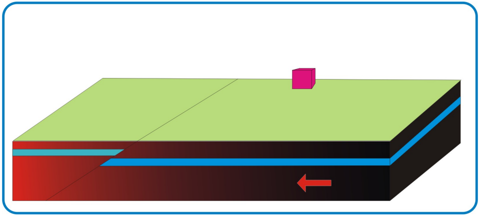

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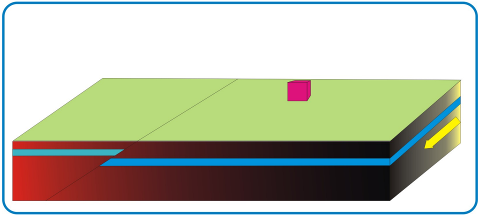

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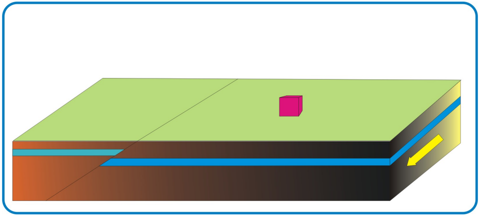

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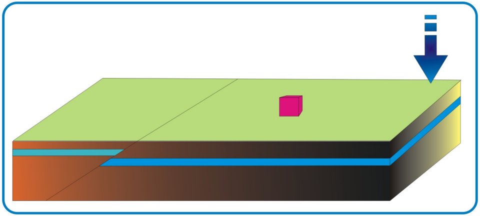

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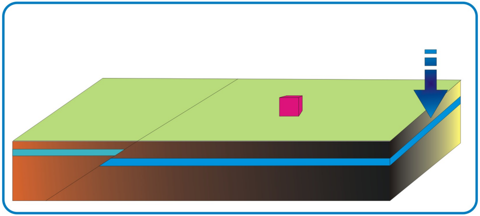

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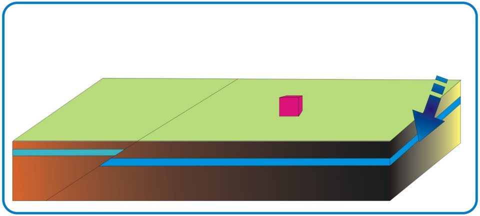

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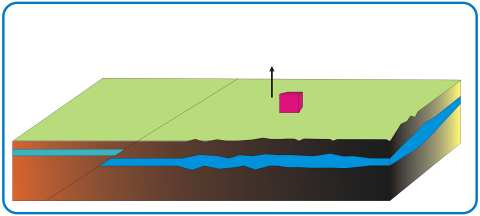

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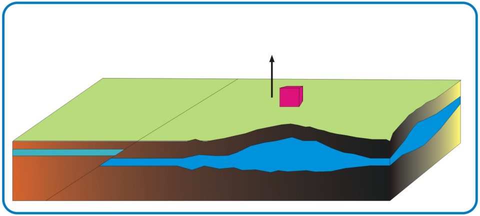

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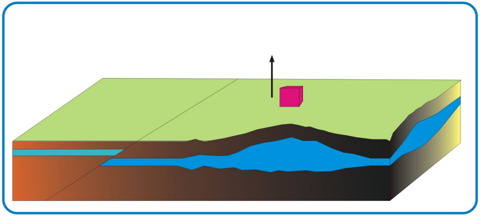

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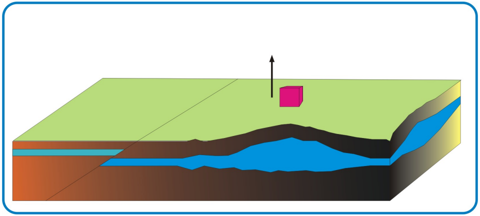

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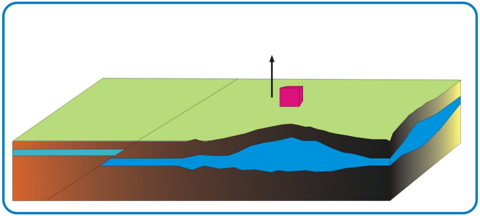

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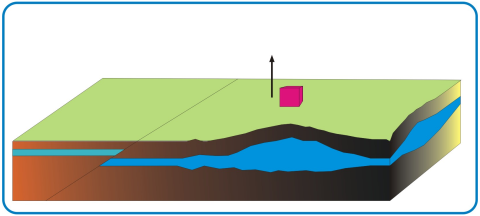

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- **Number of stations**: 318 stations ($\equiv$ 954 scalar time series).
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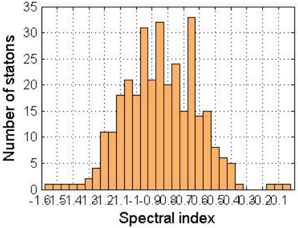

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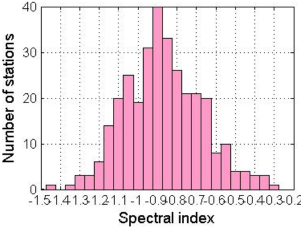
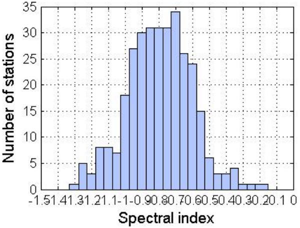


HEIGHT



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Component	μ	σ	IC_σ	Covariance Component
East (**E**)	-0.8427$\pm$0.0253	0.2214	$[0.2129, 0.2488]$	**Less noisy**
North (**N**)	-0.8648$\pm$0.0227	0.2054	$[0.1906, 0.2227]$	
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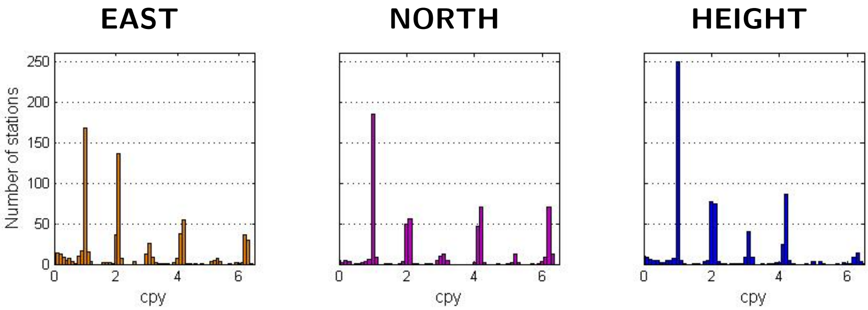

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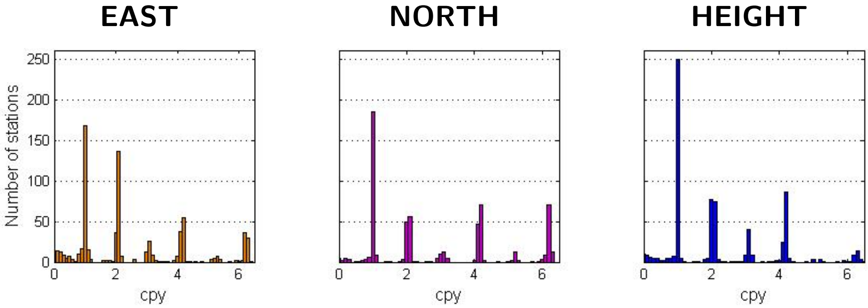

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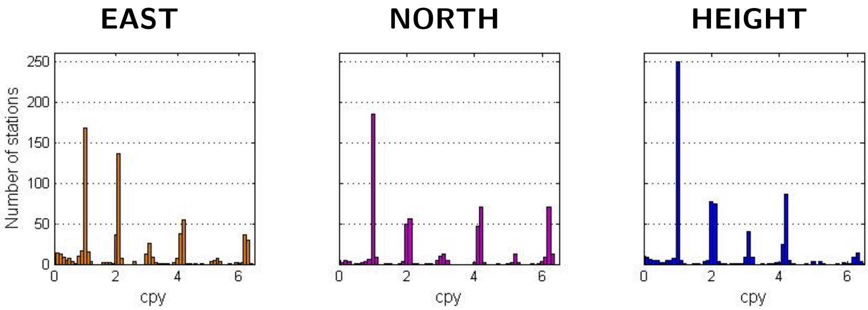

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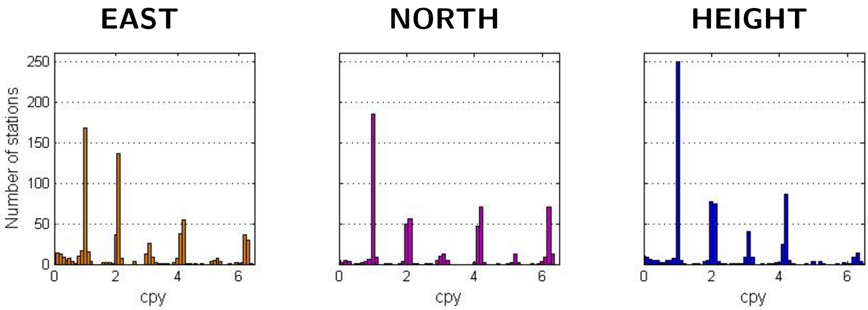

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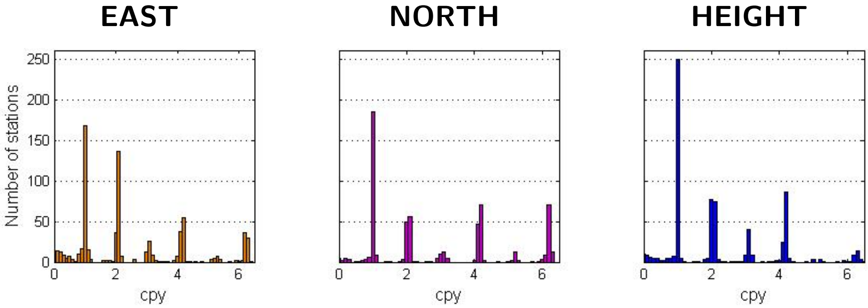

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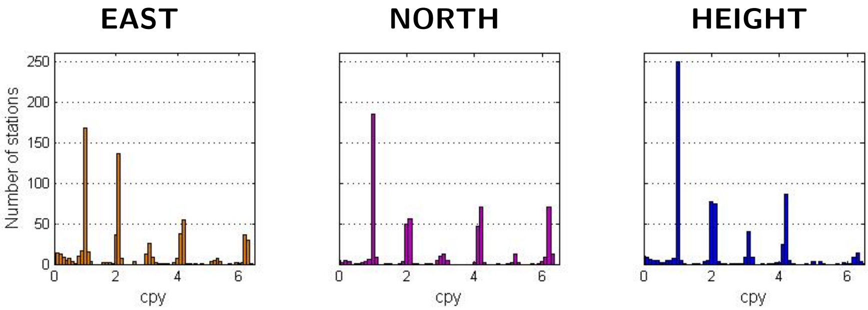

freq (cpy)	E (%)	N (%)	H (%)
0.95-1.05	52.83	58.18	78.62
2.05-2.15	43.08	17.61	23.58
3.05-3.15	8.176	4.088	12.58
4.15-4.25	17.30	22.33	27.36
6.15-6.25	11.32	22.01	2.516

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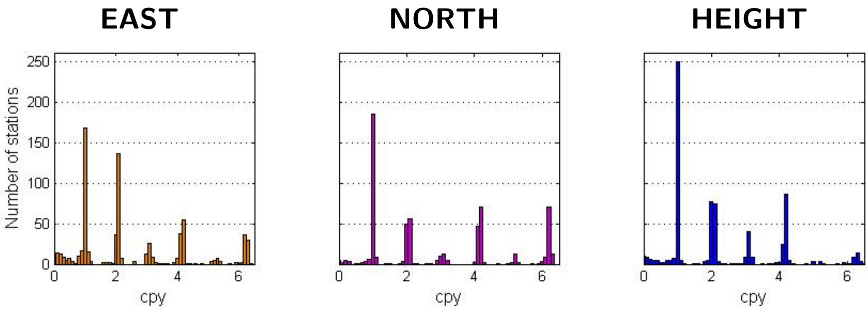

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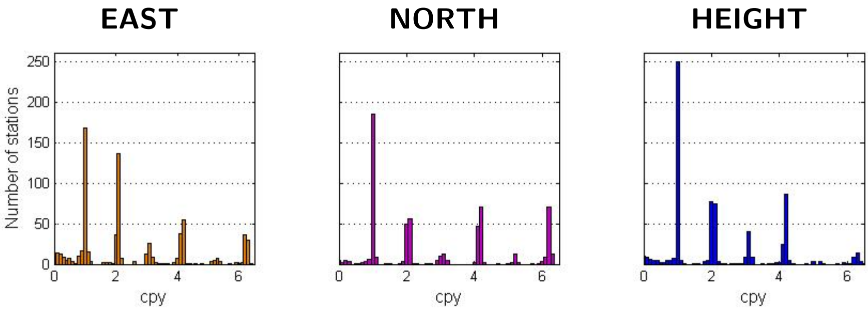

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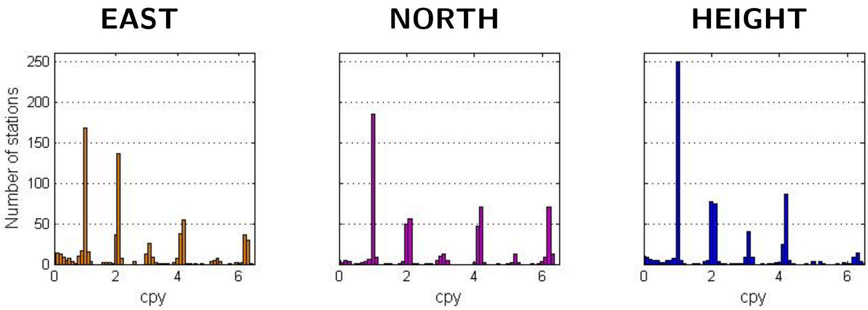

freq (cpy)	E (%)	N (%)	H (%)
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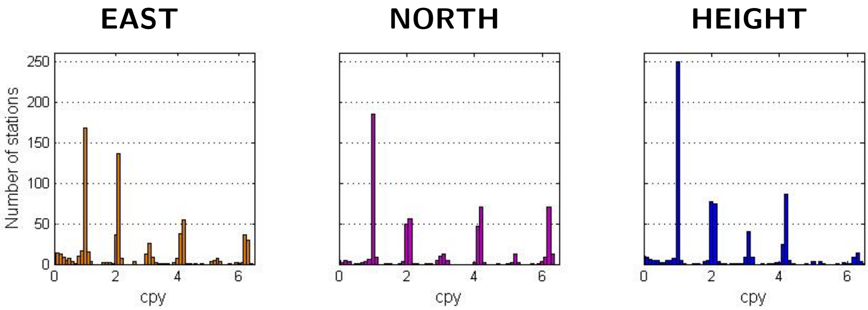
5.3. Application: Position time series of GPS stations



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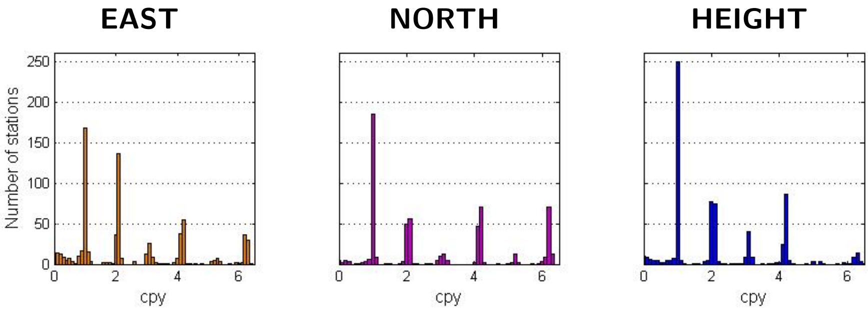

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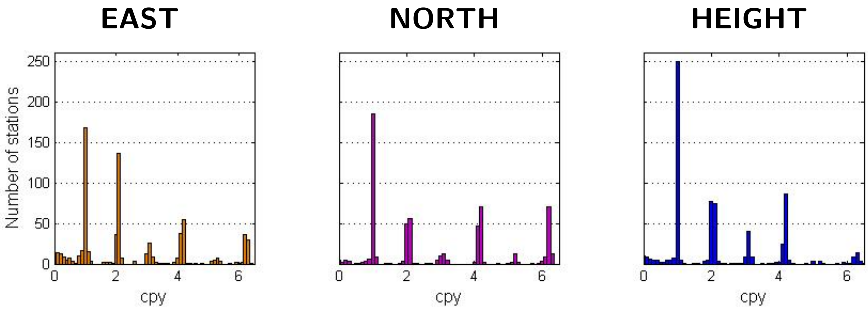
4.15-4.25	17.30	22.33	27.36
6.15-6.25	11.32	22.01	2.516

freq (cpy)	E (mm)	N (mm)	H (mm)
0.95-1.05	1.2323	1.3590	4.1446
2.05-2.15	0.5191	0.4790	1.4870
4.15-4.25	0.3132	0.2994	1.1341
6.15-6.25	0.2654	0.4325	1.0388



freq (cpy)	E (%)	N (%)	H (%)
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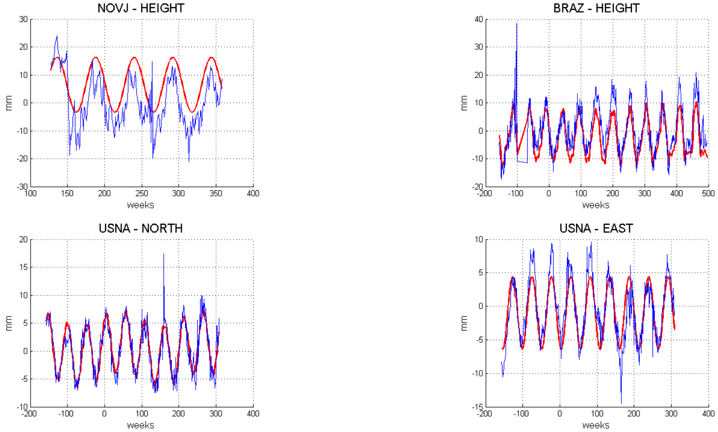
4.15-4.25	0.3132	0.2994	1.1341
6.15-6.25	0.2654	0.4325	1.0388

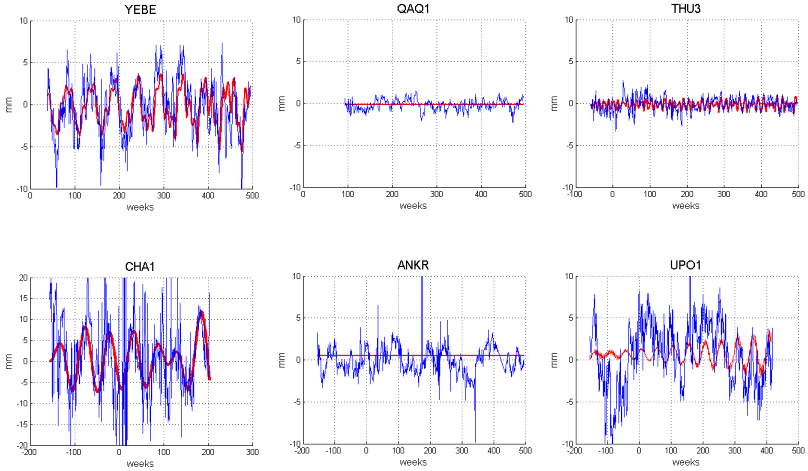
LARGER ANNUAL AMPLITUDE

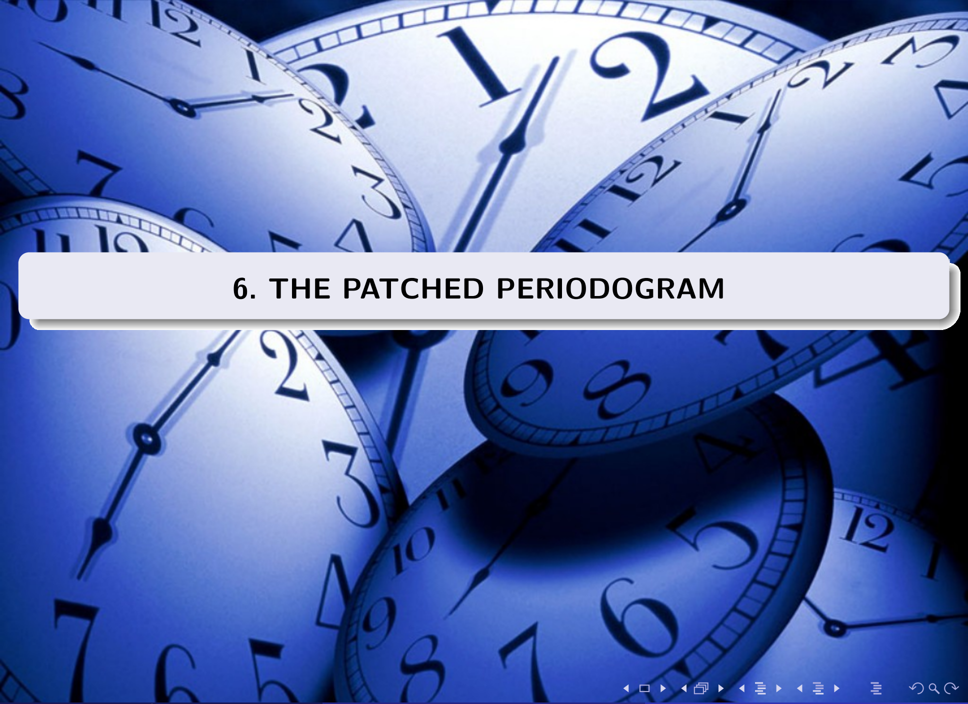


HEIGHT | **EAST** | **NORTH**

LESS **NOISY**

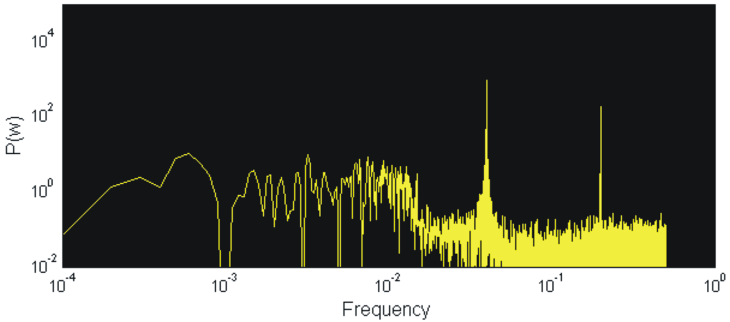
MORE **NOISY**



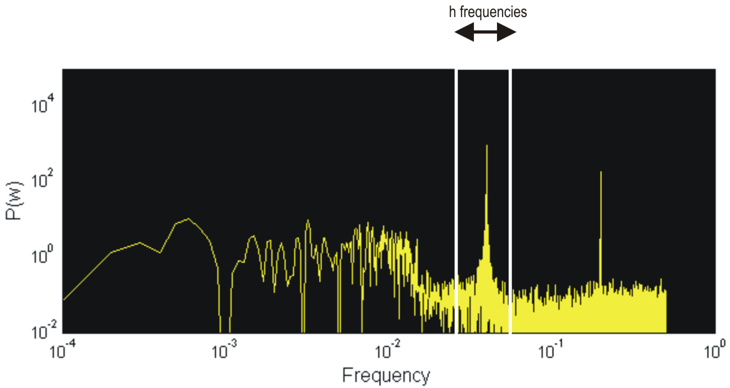


6. THE PATCHED PERIODOGRAM

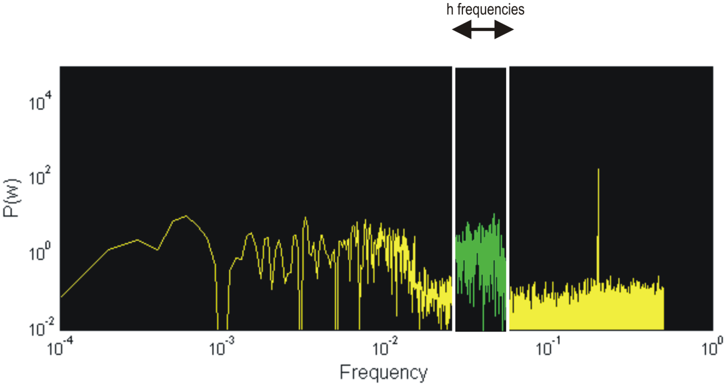
6. Patched periodogram



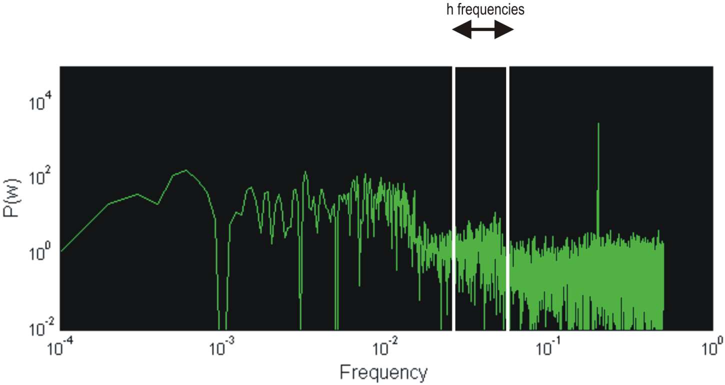
6. Patched periodogram



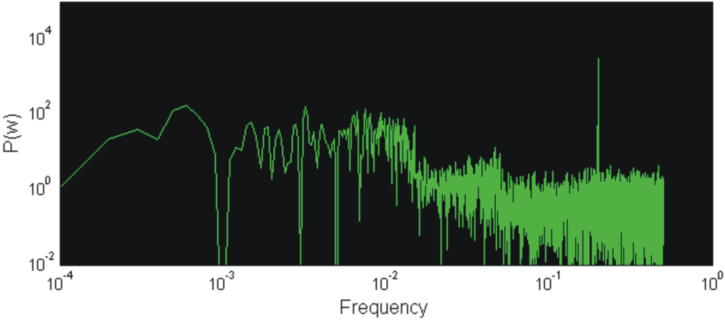
6. Patched periodogram



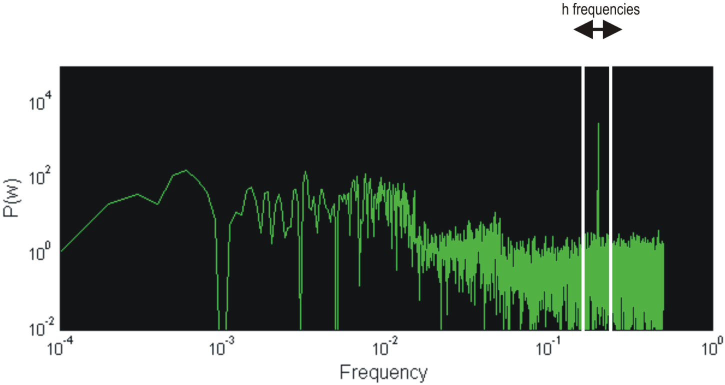
6. Patched periodogram



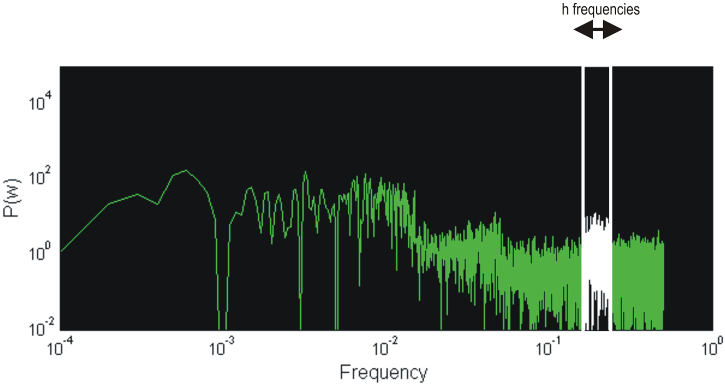
6. Patched periodogram



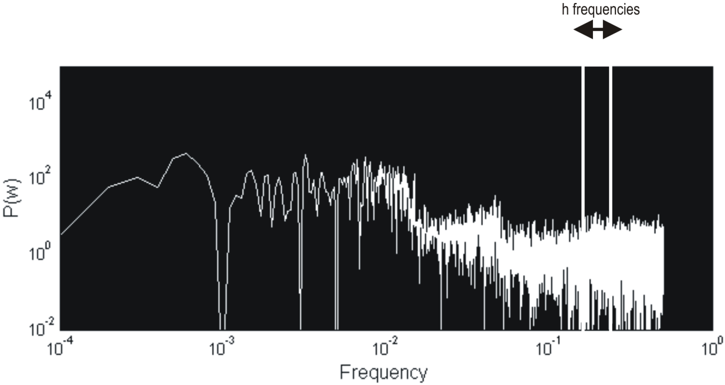
6. Patched periodogram



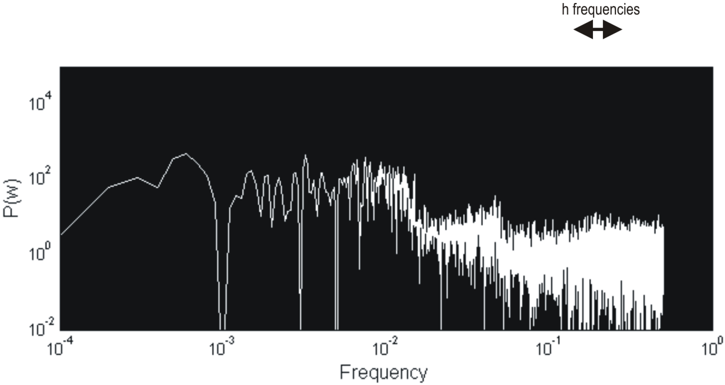
6. Patched periodogram



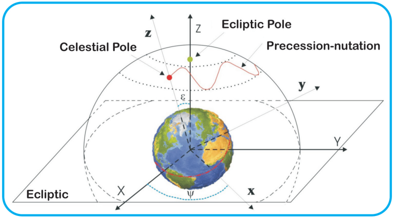
6. Patched periodogram



6. Patched periodogram

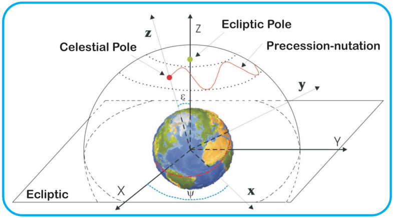


6.1. Application: IAU1980 Pole offsets



INPUT DATA

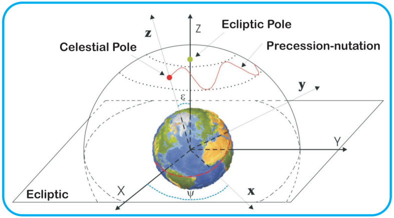
- **IAU1980** pole offsets time series



INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$

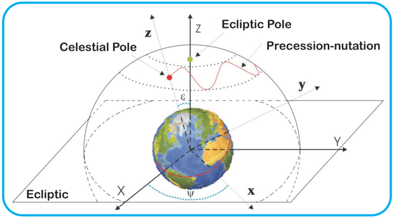
6.1. Application: IAU1980 Pole offsets



INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$
- **Time domain**: 23/Sep/2000-23/Sep/2010

6.1. Application: IAU1980 Pole offsets



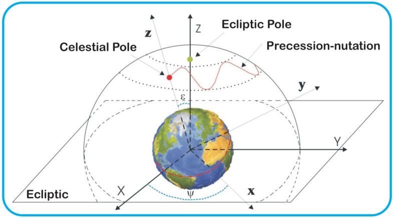
INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$
- **Time domain**: 23/Sep/2000-23/Sep/2010

ANALYSIS

- Comparison of the results by using the **common periodogram** (WNLHA) and the **patched periodogram** (QWNLH)

6.1. Application: IAU1980 Pole offsets



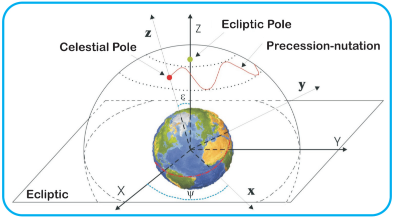
INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$
- **Time domain**: 23/Sep/2000-23/Sep/2010

ANALYSIS

- Comparison of the results by using the **common periodogram** (WNLHA) and the **patched periodogram** (QWNLH)
- **Number of frequencies**: 20

6.1. Application: IAU1980 Pole offsets

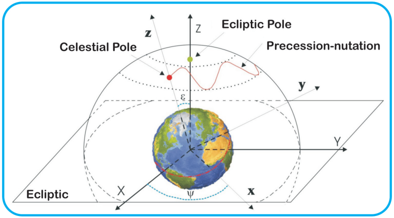


INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$
- **Time domain**: 23/Sep/2000-23/Sep/2010

ANALYSIS

- Comparison of the results by using the **common periodogram** (WNLHA) and the **patched periodogram** (QWNLH)
- **Number of frequencies**: 20
- **Window-size**: 300 frequencies



INPUT DATA

- **IAU1980** pole offsets time series
- Vectorial, evenly, daily spaced time series $(\delta\psi, \delta\epsilon)$
- **Time domain**: 23/Sep/2000-23/Sep/2010

ANALYSIS

- Comparison of the results by using the **common periodogram** (WNLHA) and the **patched periodogram** (QWNLH)
- **Number of frequencies**: 20
- **Window-size**: 300 frequencies
- **Correction stage** after 4 patches

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
4	6	1597.20	− 0.11	− 0.22	0.06	− 0.21
5	5	449.79	0.36	0.18	− 0.10	0.14
6	4	13.65	− 0.12	− 0.47	− 0.24	0.13
7	7	31.90	− 0.19	− 0.12	− 0.00	− 0.03
8	8	27.53	0.15	− 0.08	0.02	0.06
9	9	27.02	−0.01	0.06	0.01	−0.10
10	11	13.59	0.01	−0.10	−0.10	0.02
11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

6.1. Application: IAU1980 Pole offsets

No.	*Ref.*	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	**1**	365.18	1.30	– 5.76	0.36	0.02
2	**2**	182.96	1.75	– 1.11	– 0.31	– 0.53
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5	**5**	449.79	0.36	0.18	– 0.10	0.14
6	**4**	13.65	– 0.12	– 0.47	– 0.24	0.13
7	**7**	31.90	– 0.19	– 0.12	– 0.00	– 0.03
8	**8**	27.53	0.15	– 0.08	0.02	0.06
9	**9**	27.02	−0.01	0.06	0.01	−0.10
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11	**10**	371.38	−0.96	0.40	1.62	−0.12
12	**12**	9.55	– 0.00	– 0.14	– 0.01	0.03
13	**14**	14.79	0.05	– 0.02	– 0.07	– 0.04
14	**13**	13.16	0.06	−0.00	−0.01	−0.07
15	**17**	206.02	−0.01	−0.05	0.07	0.03
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6.1. Application: IAU1980 Pole offsets

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5	5	449.79	0.36	0.18	− 0.10	0.14
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11	10	371.38	−0.96	0.40	1.62	−0.12
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13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
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15	17	206.02	−0.01	−0.05	0.07	0.03
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6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
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6	4	13.65	$-$ 0.12	$-$ 0.47	$-$ 0.24	0.13
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12	12	9.55	$-$ 0.00	$-$ 0.14	$-$ 0.01	0.03
13	14	14.79	0.05	$-$ 0.02	$-$ 0.07	$-$ 0.04
14	13	13.16	0.06	-0.00	-0.01	-0.07
15	17	206.02	-0.01	-0.05	0.07	0.03
16	16	9.12	0.08	0.01	$-$ 0.04	0.03
17	15	29.78	$-$ 0.02	0.00	$-$ 0.01	$-$ 0.00

OBSERVATIONS

- Annual

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
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5	5	449.79	0.36	0.18	− 0.10	0.14
6	4	13.65	− 0.12	− 0.47	− 0.24	0.13
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8	8	27.53	0.15	− 0.08	0.02	0.06
9	9	27.02	−0.01	0.06	0.01	−0.10
10	11	13.59	0.01	−0.10	−0.10	0.02
11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

OBSERVATIONS

- Annual
- Semiannual

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	– 5.76	0.36	0.02
2	2	182.96	1.75	– 1.11	– 0.31	– 0.53
3	3	3399.98	0.54	1.40	– 0.53	1.14
4	6	1597.20	– 0.11	– 0.22	0.06	– 0.21
5	5	449.79	0.36	0.18	– 0.10	0.14
6	4	13.65	– 0.12	– 0.47	– 0.24	0.13
7	7	31.90	– 0.19	– 0.12	– 0.00	– 0.03
8	8	27.53	0.15	– 0.08	0.02	0.06
9	9	27.02	–0.01	0.06	0.01	–0.10
10	11	13.59	0.01	–0.10	–0.10	0.02
11	10	371.38	–0.96	0.40	1.62	–0.12
12	12	9.55	– 0.00	– 0.14	– 0.01	0.03
13	14	14.79	0.05	– 0.02	– 0.07	– 0.04
14	13	13.16	0.06	–0.00	–0.01	–0.07
15	17	206.02	–0.01	–0.05	0.07	0.03
16	16	9.12	0.08	0.01	– 0.04	0.03
17	15	29.78	– 0.02	0.00	– 0.01	– 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
4	6	1597.20	− 0.11	− 0.22	0.06	− 0.21
5	5	449.79	0.36	0.18	− 0.10	0.14
6	4	13.65	− 0.12	− 0.47	− 0.24	0.13
7	7	31.90	− 0.19	− 0.12	− 0.00	− 0.03
8	8	27.53	0.15	− 0.08	0.02	0.06
9	9	27.02	−0.01	0.06	0.01	−0.10
10	11	13.59	0.01	−0.10	−0.10	0.02
11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	– 5.76	0.36	0.02
2	2	182.96	1.75	– 1.11	– 0.31	– 0.53
3	3	3399.98	0.54	1.40	– 0.53	1.14
4	6	1597.20	– 0.11	– 0.22	0.06	– 0.21
5	5	449.79	0.36	0.18	– 0.10	0.14
6	4	13.65	– 0.12	– 0.47	– 0.24	0.13
7	7	31.90	– 0.19	– 0.12	– 0.00	– 0.03
8	8	27.53	0.15	– 0.08	0.02	0.06
9	9	27.02	–0.01	0.06	0.01	–0.10
10	11	13.59	0.01	–0.10	–0.10	0.02
11	10	371.38	–0.96	0.40	1.62	–0.12
12	12	9.55	– 0.00	– 0.14	– 0.01	0.03
13	14	14.79	0.05	– 0.02	– 0.07	– 0.04
14	13	13.16	0.06	–0.00	–0.01	–0.07
15	17	206.02	–0.01	–0.05	0.07	0.03
16	16	9.12	0.08	0.01	– 0.04	0.03
17	15	29.78	– 0.02	0.00	– 0.01	– 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	– 5.76	0.36	0.02
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11	10	371.38	−0.96	0.40	1.62	−0.12
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16	16	9.12	0.08	0.01	– 0.04	0.03
17	15	29.78	– 0.02	0.00	– 0.01	– 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month
- 445.5-day, **Free Core Nutation**

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	– 5.76	0.36	0.02
2	2	182.96	1.75	– 1.11	– 0.31	– 0.53
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7	7	31.90	– 0.19	– 0.12	– 0.00	– 0.03
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12	12	9.55	– 0.00	– 0.14	– 0.01	0.03
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14	13	13.16	0.06	–0.00	–0.01	–0.07
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16	16	9.12	0.08	0.01	– 0.04	0.03
17	15	29.78	– 0.02	0.00	– 0.01	– 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month
- 445.5-day, **Free Core Nutation**
- 4.3-year

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
4	6	1597.20	− 0.11	− 0.22	0.06	− 0.21
5	5	449.79	0.36	0.18	− 0.10	0.14
6	4	13.65	− 0.12	− 0.47	− 0.24	0.13
7	7	31.90	− 0.19	− 0.12	− 0.00	− 0.03
8	8	27.53	0.15	− 0.08	0.02	0.06
9	9	27.02	−0.01	0.06	0.01	−0.10
10	11	13.59	0.01	−0.10	−0.10	0.02
11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month
- 445.5-day, **Free Core Nutation**
- 4.3-year
- Periods close to a **sinodic month**

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
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8	8	27.53	0.15	− 0.08	0.02	0.06
9	9	27.02	−0.01	0.06	0.01	−0.10
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11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month
- 445.5-day, **Free Core Nutation**
- 4.3-year
- Periods close to a **sinodic month**, a **fortnight**

6.1. Application: IAU1980 Pole offsets

No.	Ref.	Π days	S_ψ mas	C_ψ mas	S_ϵ mas	C_ϵ mas
1	1	365.18	1.30	− 5.76	0.36	0.02
2	2	182.96	1.75	− 1.11	− 0.31	− 0.53
3	3	3399.98	0.54	1.40	− 0.53	1.14
4	6	1597.20	− 0.11	− 0.22	0.06	− 0.21
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9	9	27.02	−0.01	0.06	0.01	−0.10
10	11	13.59	0.01	−0.10	−0.10	0.02
11	10	371.38	−0.96	0.40	1.62	−0.12
12	12	9.55	− 0.00	− 0.14	− 0.01	0.03
13	14	14.79	0.05	− 0.02	− 0.07	− 0.04
14	13	13.16	0.06	−0.00	−0.01	−0.07
15	17	206.02	−0.01	−0.05	0.07	0.03
16	16	9.12	0.08	0.01	− 0.04	0.03
17	15	29.78	− 0.02	0.00	− 0.01	− 0.00

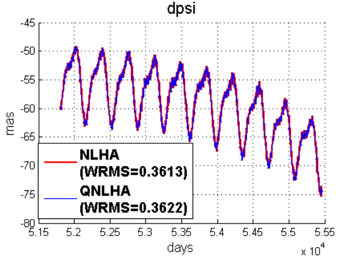
OBSERVATIONS

- Annual
- Semiannual
- 9.3-year period (Lunar perigee)
- Lunar anomalistic month
- Lunar semi-sidereal month
- 445.5-day, **Free Core Nutation**
- 4.3-year
- Periods close to a **sinodic month**, a **fortnight** and **9 days**.

6.1. Application: IAU1980 Pole offsets

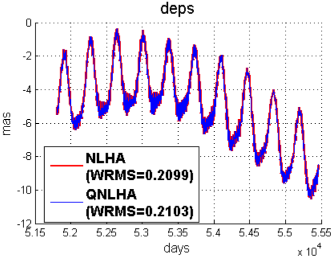
No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	-0.21	0.06	-0.26	-0.14
11	371.38	-0.25	-0.33	0.12	0.60
17	29.78	-0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	-0.02

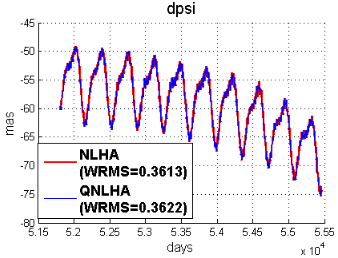
No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	−0.21	0.06	−0.26	−0.14
11	371.38	−0.25	−0.33	0.12	0.60
17	29.78	−0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	−0.02



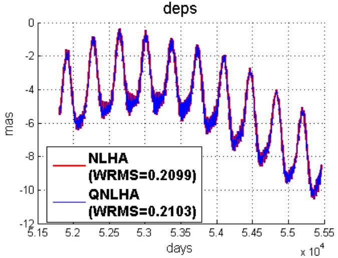
6.1. Application: IAU1980 Pole offsets

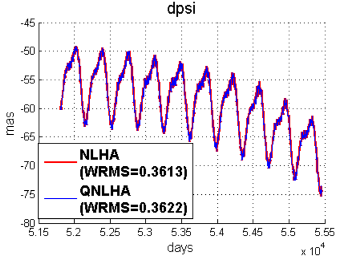
No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	−0.21	0.06	−0.26	−0.14
11	371.38	−0.25	−0.33	0.12	0.60
17	29.78	−0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	−0.02





No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	-0.21	0.06	-0.26	-0.14
11	371.38	-0.25	-0.33	0.12	0.60
17	29.78	-0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	-0.02



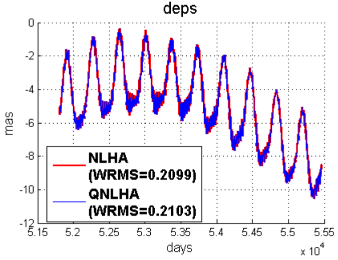


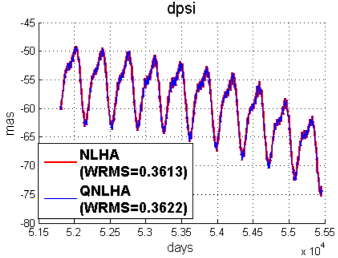
Computation time

- WNLHA:

6.1. Application: IAU1980 Pole offsets

No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	-0.21	0.06	-0.26	-0.14
11	371.38	-0.25	-0.33	0.12	0.60
17	29.78	-0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	-0.02

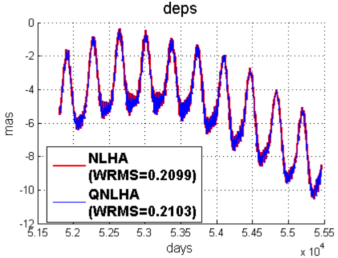


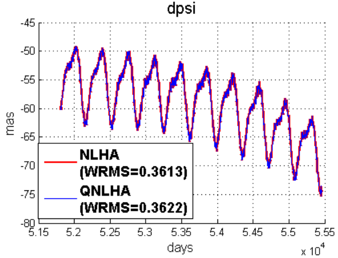


Computation time

- WNLHA: **3032.30** s

No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	-0.21	0.06	-0.26	-0.14
11	371.38	-0.25	-0.33	0.12	0.60
17	29.78	-0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	-0.02

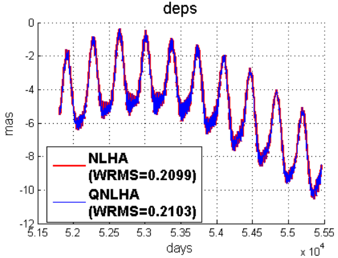


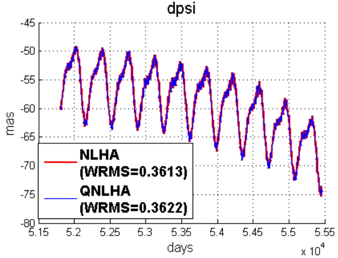


Computation time

- WNLHA: **3032.30** s
- QWNLH:

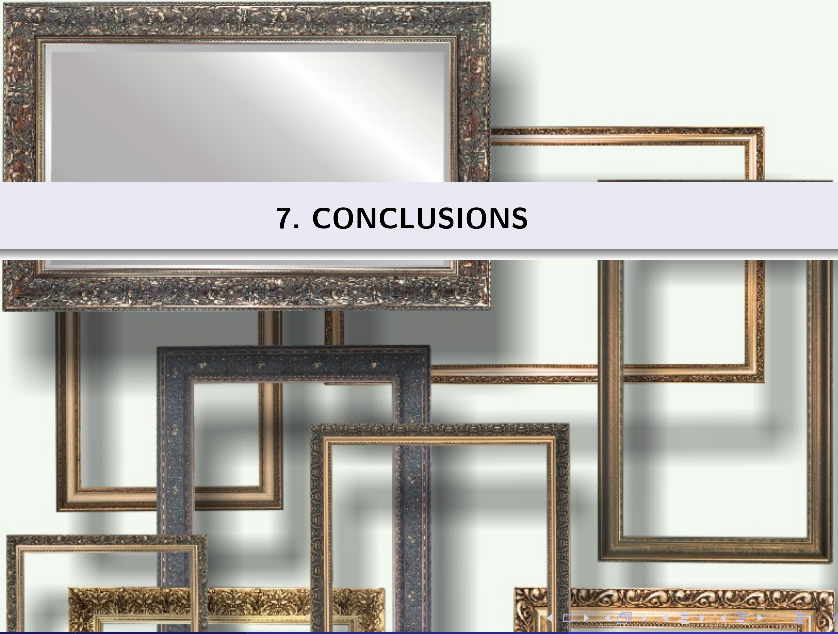
No.	Π days	SS_ψ μas	CC_ψ μas	SS_ϵ μas	CC_ϵ μas
4	1597.20	-0.21	0.06	-0.26	-0.14
11	371.38	-0.25	-0.33	0.12	0.60
17	29.78	-0.10	0.03	0.04	0.03
20	9.33	0.02	0.10	0.03	-0.02





Computation time

- WNLHA: **3032.30** s
- QWNLH: **2286.61** s



7. CONCLUSIONS

7. Conclusions

1. **Development of the FHAST algorithm** for the analysis of time series

7. Conclusions

1. **Development of the FHAST algorithm** for the analysis of time series
2. **Development of the Patched Periodogram**

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1. **Development of the FHAST algorithm** for the analysis of time series
2. **Development of the Patched Periodogram**
3. Comprehensive **software** for non-linear harmonic analysis of time series (MATLAB)

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1. **Development of the FHAST algorithm** for the analysis of time series
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COMMON PATTERNS OF ALL PACKAGES

- **Evenly or unevenly** time series with **graphical** displays

7. Conclusions

1. **Development of the FHAST algorithm** for the analysis of time series
2. **Development of the Patched Periodogram**
3. Comprehensive **software** for non-linear harmonic analysis of time series (MATLAB)

COMMON PATTERNS OF ALL PACKAGES

- **Evenly or unevenly** time series with **graphical** displays
- **Functional model**: Quadratic polynomial trend+ Fourier + mixed secular terms

7. Conclusions

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2. **Development of the Patched Periodogram**
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- **Functional model**: Quadratic polynomial trend+ Fourier + mixed secular terms
- Estimated **uncertainties** for linear and non-linear parameters

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NLHA package

- **Scalar** and **vectorial** time series with **white noise** component

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- **Scalar** and **vectorial** time series with **white noise** component
- Harmonic content can be **derived** or **fixed**

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WNLHA package (as extension of NLHA package)

- It can perform a **weighted** harmonic analysis

7. Conclusions

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WNLHA package (as extension of NLHA package)

- It can perform a **weighted** harmonic analysis
- Adds other stop criteria and includes a filtering SNR process

7. Conclusions

1. **Development of the FHAST algorithm** for the analysis of time series
2. **Development of the Patched Periodogram**
3. Comprehensive **software** for non-linear harmonic analysis of time series (MATLAB)

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- **Evenly or unevenly** time series with **graphical** displays
- **Functional model**: Quadratic polynomial trend+ Fourier + mixed secular terms
- Estimated **uncertainties** for linear and non-linear parameters

NLHA package

- **Scalar** and **vectorial** time series with **white noise** component
- Harmonic content can be **derived** or **fixed**

WNLHA package (as extension of NLHA package)

- It can perform a **weighted** harmonic analysis
- Adds other stop criteria and includes a filtering SNR process
- Periodogram estimation can be slightly accelerated

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated

7. Conclusions

FHAST package:

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- (Co)variance component estimations can be negative

7. Conclusions

FHAST package:

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- (Co)variance component estimations can be negative

RFHAST package (extension of FHAST package)

- An **exponential re-parametrization** is performed for the LS-VCE process

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated
- (Co)variance component estimations can be negative

RFHAST package (extension of FHAST package)

- An **exponential re-parametrization** is performed for the LS-VCE process

ANALYZED TIME SERIES

- Celestial Pole offsets

7. Conclusions

FHAST package:

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RFHAST package (extension of FHAST package)

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ANALYZED TIME SERIES

- Celestial Pole offsets
- Length of day variations

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated
- (Co)variance component estimations can be negative

RFHAST package (extension of FHAST package)

- An **exponential re-parametrization** is performed for the LS-VCE process

ANALYZED TIME SERIES

- Celestial Pole offsets
- Length of day variations
- Geocenter variations caused by continental water flux

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated
- (Co)variance component estimations can be negative

RFHAST package (extension of FHAST package)

- An **exponential re-parametrization** is performed for the LS-VCE process

ANALYZED TIME SERIES

- Celestial Pole offsets
- Length of day variations
- Geocenter variations caused by continental water flux
- Multidimensional spatio-temporal models for continental water flux

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated
- (Co)variance component estimations can be negative

RFHAST package (extension of FHAST package)

- An **exponential re-parametrization** is performed for the LS-VCE process

ANALYZED TIME SERIES

- Celestial Pole offsets
- Length of day variations
- Geocenter variations caused by continental water flux
- Multidimensional spatio-temporal models for continental water flux
- Position of GPS stations

7. Conclusions

FHAST package:

- Addressed to **scalar** time series+ **colored** noise
- **Stochastic Model** that can be fixed or estimated
- (Co)variance component estimations can be negative

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  - Improve the **computation time** $\Rightarrow$ **PATCHED PERIODOGRAM**

ARTICLES AND PAPERS

1. Martinez-Ortiz P. and Ferrandiz J.M., **Linear versus non-linear harmonic method in MATLAB for the spatio-temporal modelling of continental water flux.** (IN PREPARATION)
2. Martinez-Ortiz P., Ferrandiz J.M., Altamimi Z. and Collilieux X., **FHAST toolbox: A combination of non-linear harmonic and stochastic models for geodetic time series analysis.** (IN PREPARATION)
3. Martinez-Ortiz P. and Ferrandiz J.M., **The patched periodogram: an new accelerated way of estimating spectral lines**(IN PREPARATION)
4. Ferrándiz J.M.; Martínez-Ortiz P.; García D., **Effects of time gravity changes on the Earth nutations** (IN PREPARATION)

CONGRESS AND MEETINGS

1. Ferrándiz J.M.; Martínez-Ortiz P.; García D., **Effects of time gravity changes on the Earth nutations**, EGU General Assembly 2011 (POSTER), Viena (AUSTRIA) (03/04/2011 - 08/04/2011)
2. Martinez-Ortiz P.; Ferrandiz J.; Altamimi Z.; Collilieux X., **Combination of Non-Linear Harmonic and Stochastic Models for Geodetic Time Series Analysis** (ORAL COMMUNICATION), 10th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2010, Almería, Spain and Madison, WI, USA (26/06/2010 - 30/06/2010)
3. Martínez-Ortiz P.; Ferrándiz J.M., **Non-linear Harmonic Models for some Geophysical Time Series** (ORAL COMMUNICATION), EGU General Assembly 2010, Viena (AUSTRIA) (02/05/2010 - 07/05/2010)
4. Martínez-Ortiz, P.A.; Ferrándiz, J.M., **Applications of non-linear harmonic analysis to some geophysical problems** (ORAL COMMUNICATION), CMMSE 2009, Gijón (30/06/2009 - 03/07/2009)

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NON-LINEAR HARMONIC MODELS FOR GEODETIC TIME SERIES

Modelos armónicos no lineales para series temporales geodéticas

Pedro Antonio Martínez Ortiz
Dept. Applied Mathematics
University of Alicante. Spain

June, 18th 2011

Memoria para la obtención del grado de Doctor

Director: José Manuel Ferrándiz Leal